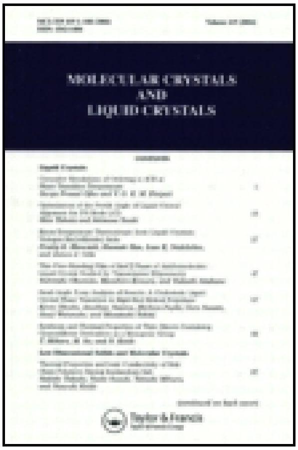

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A Continuum Theory for Smectic C Liquid Crystals

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1. INTRODUCTION

Our aim in this paper is to present a rather general non-linear continuum theory that may prove useful for the interpretation of observations and experiments upon smectic C liquid crystals. In its present formulation our theory is constrained to exclude variations in the layer spacing thickness and also changes in tilt with respect to the layer normal. De Gennes[1] discusses a theory of this type for smectic A liquid crystals, but for these liquids applications of such a constrained theory are somewhat more restricted than for smectic C materials, and consequently our interest focuses on the latter. Throughout, much of our motivation and reasoning stem from experience with nematic theory, details of which are available in, for example, the reviews by Ericksen[2] and Leslie[3] as well as the books by de Gennes[1] and Chandrasekhar.[4]

Our description of smectic configurations follows de Gennes[1] and employs two directors, one normal to the layers and the second tangential to the layers. It turns out that with the symmetries assumed the local energy function quadratic in director gradients has essentially nine terms, and is identical to that proposed by the Orsay Group[5] when restricted to small perturbations of planar layers. While there is more than one option for the derivation of equilibrium equations,[6] our approach here adopts a principle of virtual work which is a generalization of that first employed by Ericksen[7] for nematics. Also, we confine our attention to non-chiral materials, but Nakagawa[8] extends our model to smectic C* liquid crystals. Notwithstanding the complexity of the resulting equations, a number of solutions in which the layers are non-planar are possible including configurations with cylindrical, spherical and toroidal layers. However, following the discussion by Bragg[9] and Frank[10] a serious test for any such static theory of smectics is that it must also predict layers forming Dupin cyclides, and the present theory proves equal to this challenge.[11] Moreover, our work[12] leads to further cyclide solutions not discussed by Bragg or Frank.

The latter part of the paper proceeds to a derivation of a dynamic theory for smectic C liquid crystals similar to that for nematics described by Leslie.[3] With the symmetries assumed the dissipative stress contains twenty terms, this after appeal to Onsager relations, but as for nematics there is no dissipative term in the couple stress. Some preliminary studies of flow problems, however, indicate that this degree of complexity is necessary to produce flow alignment in certain configurations, and also predict the likelihood of transverse flows in other arrangements. Calculations presently in progress for light scattering will allow comparison with the earlier linear theory by Martin, Parodi and Pershan.[13] Finally, our dynamic theory differs in certain important respects from that proposed by Schiller.[14]

Whenever possible vector notation is employed, but inevitably we must also use Cartesian tensor notation for tensorial quantities. In the latter a repeated index denotes a summation, a comma preceding a suffix a partial derivative, and δ_{ij} and e_{ijk} the Kronecker delta and alternator, respectively. A superposed dot represents a material time derivative.

2. STATICS

Theory

To describe the layered structure in a smectic C liquid crystal, it is convenient to employ a density wave vector $\mathbf{a}$, which in the absence of defects is subject to the constraint[15]

$$\operatorname{curl} \mathbf{a} = 0. \tag{2.1}$$

However, since our theory assumes that the layers remain of constant thickness, we can without loss of generality simply regard $\mathbf{a}$ as a unit vector identical to the layer normal. Following de Gennes[1] a further unit vector $\mathbf{c}$ describes the direction of tilt of the alignment with respect to the layer normal, this second vector being perpendicular to the layer normal $\mathbf{a}$. Hence the two directors in our constrained theory are subject to the additional constraints

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{c} = 1, \quad \mathbf{a} \cdot \mathbf{c} = 0, \tag{2.2}$$

which with (2.1) give rise to undetermined multipliers in our equations.

As in nematic theory the local energy density is assumed to be of the form

$$W = W(\mathbf{a}, \mathbf{c}, \nabla\mathbf{a}, \nabla\mathbf{c}), \tag{2.3}$$

being a quadratic function of the gradients. Here the non-chiral smectic symmetry dictates that the above dependence is isotropic, and moreover invariant to the simultaneous changes of sign

$$\mathbf{a} \rightarrow -\mathbf{a} \quad \text{and} \quad \mathbf{c} \rightarrow -\mathbf{c}. \tag{2.4}$$

In this event the quadratic energy function can be written as

$$\begin{aligned} 2W = {} & K_1(\nabla \cdot \mathbf{a})^2 + K_2(\nabla \cdot \mathbf{c})^2 + K_3(\mathbf{a} \cdot \nabla \times \mathbf{c})^2 \\ & + K_4(\mathbf{c} \cdot \nabla \times \mathbf{c})^2 + K_5(\mathbf{b} \cdot \nabla \times \mathbf{c})^2 + 2K_6(\nabla \cdot \mathbf{a})(\mathbf{b} \cdot \nabla \times \mathbf{c}) \\ & + 2K_7(\mathbf{a} \cdot \nabla \times \mathbf{c})(\mathbf{c} \cdot \nabla \times \mathbf{c}) + 2K_8(\nabla \cdot \mathbf{c})(\mathbf{b} \cdot \nabla \times \mathbf{c}) \\ & + 2K_9(\nabla \cdot \mathbf{a})(\nabla \cdot \mathbf{c}) \end{aligned} \tag{2.5}$$

where $\mathbf{b} = \mathbf{a} \times \mathbf{c}$, surface terms being omitted. There are, however, alternative yet equivalent expressions for this energy, but we do not discuss this here (see Reference 16). For small perturbations the above is identical to the expression given earlier by the Orsay group.[5]

Our derivation of the equilibrium equations[6] parallels that by Ericksen[7] for nematics, assuming a principle of virtual work for a given volume V of liquid crystal as follows

$$\delta \int_V W dv = \int_V (\mathbf{F} \cdot \delta\mathbf{x} + \mathbf{G}^a \cdot \Delta\mathbf{a} + \mathbf{G}^c \cdot \Delta\mathbf{c})\, dv + \int_{\partial V} (\mathbf{t} \cdot \delta\mathbf{x} + \mathbf{s}^a \cdot \Delta\mathbf{a} + \mathbf{s}^c \cdot \Delta\mathbf{c})\, ds \tag{2.6}$$

where

$$\Delta\mathbf{a} = \delta\mathbf{a} + (\delta\mathbf{x} \cdot \text{grad})\mathbf{a}, \quad \Delta\mathbf{c} = \delta\mathbf{c} + (\delta\mathbf{x} \cdot \text{grad})\mathbf{c}, \tag{2.7}$$

and $\mathbf{F}$ denotes external body force per unit volume, $\mathbf{G}^a$ and $\mathbf{G}^c$ generalized external body forces per unit volume, $\mathbf{t}$ surface traction per unit area, and $\mathbf{s}^a$ and $\mathbf{s}^c$ generalized surface tractions per unit area. As in Ericksen's derivation, the virtual displacements $\delta\mathbf{x}$ are subject to

$$\text{div}\, \delta\mathbf{x} = 0, \tag{2.8}$$

on account of the assumed incompressibility. The identity

$$e_{ijk}(\Delta a_{k,j} - a_{k,p}\delta x_{p,j}) = 0 \tag{2.9}$$

is also available. Following essentially the same manipulations as Ericksen, one can express the surface forces in terms of corresponding stress tensors and the unit surface normal $\mathbf{v}$ in the forms

$$t_i = t_{ij}\nu_j, \quad s_i^a = \alpha a_i + \beta c_i + s_{ij}^a \nu_j, \quad s_i^c = \gamma c_i + \beta a_i + s_{ij}^c \nu_j, \tag{2.10}$$

where

$$\left.\begin{aligned} t_{ij} &= -p\delta_{ij} + \beta_p e_{pjk} a_{k,i} - \frac{\partial W}{\partial a_{k,j}} a_{k,i} - \frac{\partial W}{\partial c_{k,j}} c_{k,i}, \\ s_{ij}^a &= e_{ijk}\beta_k + \frac{\partial W}{\partial a_{i,j}}, \quad s_{ij}^c = \frac{\partial W}{\partial c_{i,j}}, \end{aligned}\right\} \tag{2.11}$$

the scalars α, β and γ arising from the constraints Equation (2.2), the pressure p from Equation (2.8), and the vector $\boldsymbol{\beta}$ from Equation (2.1). In addition, the virtual work postulate Equation (2.6) leads to the following balance laws

$$F_i + t_{ij,j} = 0, \tag{2.12}$$

and

$$\left.\begin{aligned} \left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j} - \frac{\partial W}{\partial a_i} + G_i^a + \lambda a_i + \mu c_i + e_{ijk}\beta_{k,j} &= 0, \\ \left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j} - \frac{\partial W}{\partial c_i} + G_i^c + \varkappa c_i + \mu a_i &= 0, \end{aligned}\right\} \tag{2.13}$$

the multipliers λ, μ and $\varkappa$ again arising from the constraints Equation (2.2). Not unexpectedly the Equation (2.12) represents a balance of forces, but less obviously perhaps Equations (2.13) are equivalent to a balance of moments (see Reference 6 for details). Indeed one can show that the body moment $\mathbf{K}$ and the couple stress tensor $\mathbf{l}$ are given by

$$\mathbf{K} = \mathbf{a} \times \mathbf{G}^a + \mathbf{c} \times \mathbf{G}^c, \tag{2.14}$$

and

$$l_{ij} = \beta_p a_p \delta_{ij} - \beta_i a_j + e_{ipq}\left(a_p \frac{\partial W}{\partial a_{q,j}} + c_p \frac{\partial W}{\partial c_{q,j}}\right), \tag{2.15}$$

respectively, essentially through a repetition of the argument by Ericksen.[7]

When external forces and moments are absent, it is relatively straightforward to combine the above Equations (2.12) and (2.13) to obtain the integral

$$p + W = p_o, \tag{2.16}$$

where p_o is an arbitrary constant. With certain assumptions regarding the external forces and couples, a similar result follows when these terms are present. Consequently the Equation (2.12) representing balance of forces need not concern us further, unless to compute forces, and we therefore concentrate upon the Equations (2.13) representing balance of couples.

Solutions

We now provide some examples of static solutions which satisfy Equation (2.13). There are essentially six types of well behaved surface which readily provide static configurations for a restricted six term version (K_1 to K_6) of the energy given by Equation (2.5). These are the Dupin (or hyperbolic) cylides, circular tori of revolution, spheres, parabolic cyclides, infinite cylinders and planes. Solutions corresponding to infinite cylinders and planes, together with any necessary Lagrange multipliers, can be derived in a straight forward manner while solutions for the remaining four types are more intricate. We shall present a brief summary of the solutions for Dupin and parabolic cyclides (see Nakagawa,[11] and Stewart *et al.*[12] for details) and mention the solutions for spheres (see Leslie *et al.*[6]).

The general equation for a Dupin cyclide in Cartesian coordinates is

$$(x^2 + y^2 + z^2 - r^2)^2 - 2A^2(1 + e^2)(x^2 + r^2) - 2A^2(1 - e^2)(y^2 - z^2) + 8A^2\, erx + A^4(1 - e^2)^2 = 0, \tag{2.17}$$

where the confocal conics enabling the construction of this surface are given by the ellipse and hyperbola

$$\left.\begin{aligned} \frac{x^2}{A^2} + \frac{y^2}{B^2} &= 1 \\ z &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{x^2}{C^2} - \frac{z^2}{B^2} &= 1 \\ y &= 0 \end{aligned}\right\}. \tag{2.18}$$

Here $C^2 = A^2 - B^2$ while in Equation (2.17) r is some real parameter and the eccentricity e is defined by $e = C/A$. Equation (2.17) can be parameterized as

$$x(A \cosh u\text{-}C \cos v) = A \cos v(A \cosh u\text{-}r) + C \cosh u(r\text{-}C \cos v),$$

$$y(A \cosh u\text{-}C \cos v) = B \sin v(A \cosh u\text{-}r),$$

$$z(A \cosh u\text{-}C \cos v) = B \sinh u(r\text{-}C \cos v), \tag{2.19}$$

where, for the surfaces of interest,

$$C \cos v \leq r \leq A \cosh u,$$

$$-\infty < u < +\infty,$$

$$0 \leq v \leq 2\pi. \tag{2.20}$$

Transforming from the (x, y, z) to the (r, u, v) frame we can show that

$$\mathbf{a} = (1, 0, 0) \quad \text{and} \quad \mathbf{c} = (0, 1, 0) \tag{2.21}$$

satisfy the constraints Equations (2.1) and (2.2) and provide a solution to the transformed version of Equations (2.13) where we can derive suitable Lagrange multipliers. Further, as mentioned by Nakagawa,[11] if e vanishes (i.e., the conics become a circle and a straight line) then we obtain the solution for the circular torus.

The Cartesian equation of a parabolic cyclide is

$$x(x^2 + y^2 + z^2) + (x^2 + y^2)(l - \mu) - z^2(l + \mu) - (x - \mu + l)(l + \mu)^2 = 0 \qquad (2.22)$$

where the confocal conics essential to the construction are the two parabolas

$$\left.\begin{aligned} y^2 &= 4l(x + l) \\ z &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} z^2 &= -4lx \\ y &= 0 \end{aligned}\right\}, \qquad (2.23)$$

and l and μ are real parameters. Equation (2.22) can be parameterized as

$$\begin{aligned} x(1 + \vartheta^2 + t^2) &= \mu(\vartheta^2 + t^2 - 1) + l(t^2 - \vartheta^2 - 1), \\ y(1 + \vartheta^2 + t^2) &= 2t(l(\vartheta^2 + 1) + \mu), \\ z(1 + \vartheta^2 + t^2) &= 2\vartheta(lt^2 - \mu), \end{aligned} \qquad (2.24)$$

where

$$\begin{aligned} -\infty &< \vartheta < +\infty, \\ -\infty &< t < +\infty, \\ -\infty &< \mu < +\infty. \end{aligned} \qquad (2.25)$$

Transforming from the Cartesian to the (μ, ϑ, t) frame we find that

$$\mathbf{a} = (1, 0, 0) \quad \text{and} \quad \mathbf{c} = (0, 1, 0) \qquad (2.26)$$

fulfil constraints (2.1), (2.2) and the transformed Equations (2.13) upon obtaining suitable Lagrange multipliers.

In spherical coordinates (r, ϑ, φ), static solutions can be found when, ignoring singularities for the present, we set

$$\mathbf{a} = (1, 0, 0) \quad \text{and} \quad \mathbf{c} = (0, 1, 0) \tag{2.27}$$

or

$$\mathbf{a} = (1, 0, 0) \quad \text{and} \quad \mathbf{c} = (0, 0, 1). \tag{2.28}$$

3. DYNAMICS

Theory

Our derivation of a dynamical theory for smectic C liquid crystals essentially employs the balance laws of classical fluid mechanics. The assumption of incompressibility requires that the velocity vector $\mathbf{v}$ satisfy

$$\text{div } \mathbf{v} = 0 \tag{3.1}$$

and thus conservation of mass simply reduces to density ρ being a constant. Balance of linear momentum is as usual

$$\rho \dot{v}_i = F_i + t_{ij,j}, \tag{3.2}$$

$\mathbf{F}$ again denoting the external body force per unit volume and $\mathbf{t}$ the stress tensor. However, our equation for balance of angular momentum takes the slightly extended form

$$0 = K_i + e_{ijk}t_{kj} + l_{ij,j} \tag{3.3}$$

with $\mathbf{K}$ the external body moment per unit volume, and $\mathbf{l}$ the couple stress tensor. Here the inertial term is assumed negligible. In addition, the ensuing derivation appeals to the rate of viscous dissipation per unit volume $\mathcal{D}$ being positive, and one can show from consideration of the rate at which work is done on an arbitrary volume that

$$\dot{W} + \mathcal{D} = t_{ij}v_{i,j} + l_{ij}\omega_{i,j} - \omega_i e_{ijk}t_{kj}, \tag{3.4}$$

where W is the stored energy Equation (2.3) and $\boldsymbol{\omega}$ the local angular velocity of the fluid element.

Given the above static theory, it seems reasonable to set

$$t_{ij} = -p\delta_{ij} + \beta_p e_{pjk} a_{k,i} - \frac{\partial W}{\partial a_{k,j}} a_{k,i} - \frac{\partial W}{\partial c_{k,j}} c_{k,i} + \tilde{t}_{ij}, \tag{3.5}$$

and

$$l_{ij} = \beta_p a_p \delta_{ij} - \beta_i a_j + e_{ipq}\left(a_p \frac{\partial W}{\partial a_{q,j}} + c_p \frac{\partial W}{\partial c_{q,j}}\right) + \tilde{l}_{ij}, \tag{3.6}$$

where $\tilde{\mathbf{t}}$ and $\tilde{\mathbf{l}}$ denote dynamic contributions, and further to assume that

$$\tilde{\mathbf{t}} \text{ and } \tilde{\mathbf{l}} \text{ are functions of } \mathbf{a}, \mathbf{c}, \boldsymbol{\omega} \text{ and } \nabla \mathbf{v}, \tag{3.7}$$

this being analogous to the derivation of nematic theory given for example by Leslie,[3] since

$$\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a} \quad \text{and} \quad \dot{\mathbf{c}} = \boldsymbol{\omega} \times \mathbf{c}. \tag{3.8}$$

Invariance and symmetry require that the dependence Equation (3.7) be replaced by

$$\tilde{\mathbf{t}} \text{ and } \tilde{\mathbf{l}} \text{ are isotropic functions of } \mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{C} \text{ and } \mathbf{D}. \tag{3.9}$$

where

$$\left.\begin{aligned} 2D_{ij} &= v_{i,j} + v_{j,i}, & 2W_{ij} &= v_{i,j} - v_{j,i}, \\ A_i &= \dot{a}_i - W_{ik} a_k, & C_i &= \dot{c}_i - W_{ik} c_k, \end{aligned}\right\} \tag{3.10}$$

and symmetry further dictates that the functions are unaltered by the transformation, Equation (2.4).

Our constitutive assumptions Equations (3.5) and (3.6) can be shown to reduce the Equation (3.4) to an inequality

$$\mathcal{D} = \tilde{t}_{ij} v_{i,j} + \tilde{l}_{ij} \omega_{i,j} - \omega_i e_{ijk} \tilde{t}_{kj} \geq 0, \tag{3.11}$$

this following by appeal to Equations (3.8), the identity

$$e_{ipq}\left(a_p \frac{\partial W}{\partial a_q} + a_{p,k} \frac{\partial W}{\partial a_{q,k}} + a_{k,p} \frac{\partial W}{\partial a_{k,q}} + c_p \frac{\partial W}{\partial c_q} + c_{p,k} \frac{\partial W}{\partial c_{q,k}} + c_{k,p} \frac{\partial W}{\partial c_{k,q}}\right) = 0, \tag{3.12}$$

which is a consequence of the invariance property of the stored energy (compare Ericksen[7]), and

$$e_{ijk}a_{k,p}v_{p,j} = a_j\omega_{i,j} - a_i\omega_{j,j} + \omega_i a_{j,j} - \omega_j a_{i,j}, \tag{3.13}$$

which can be derived from the constraint Equation (2.1) using the first of Equations (3.8). The nature of the inequality Equation (3.11) and our constitutive assumptions Equations (3.7) immediately yield

$$\tilde{l}_{ij} = 0, \tag{3.14}$$

analogous to the result for a nematic.

Somewhat naturally we here assume that the dissipative stress is linear in the vectors **A** and **C** and the rate of strain tensor **D** but this leads to an expression with thirty two terms. However, appeal to Onsager relations reduces this considerably to twenty terms. In this way, employing the notation

$$D_i^a = D_{ij}a_j \quad \text{and} \quad D_i^c = D_{ij}c_j, \tag{3.15}$$

we obtain for the symmetric part of the dynamic stress

$$\begin{aligned}
\tilde{t}_{ij}^s = {} & \mu_0 D_{ij} + \mu_1 a_p D_p^a a_i a_j + \mu_2(D_i^a a_j + D_j^a a_i) + \mu_3 c_p D_p^c c_i c_j \\
& + \mu_4(D_i^c c_j + D_j^c c_i) + \mu_5 c_p D_p^a(a_i c_j + a_j c_i) \\
& + \lambda_1(A_i a_j + A_j a_i) \\
& + \lambda_2(C_i c_j + C_j c_i) + \lambda_3 c_p A_p(a_i c_j + a_j c_i) \\
& + \varkappa_1(D_i^a c_j + D_j^a c_i + D_i^c a_j + D_j^c a_i) \\
& + \varkappa_2[a_p D_p^a(a_i c_j + a_j c_i) + 2a_p D_p^c a_i a_j] \\
& + \varkappa_3[c_p D_p^c(a_i c_j + a_j c_i) + 2a_p D_p^c c_i c_j] \\
& + \tau_1(C_i a_j + C_j a_i) + \tau_2(A_i c_j + A_j c_i) \\
& + 2\tau_3 c_p A_p a_i a_j + 2\tau_4 c_p A_p c_i c_j,
\end{aligned} \tag{3.16}$$

and for the skew symmetric part

$$\begin{aligned}
\tilde{t}_{ij}^{ss} = {} & \lambda_1(D_j^a a_i - D_i^a a_j) + \lambda_2(D_j^c c_i - D_i^c c_j) + \lambda_3 c_p D_p^a(a_i c_j - a_j c_i) \\
& + \lambda_4(A_j a_i - A_i a_j) + \lambda_5(C_j c_i - C_i c_j) + \lambda_6 c_p A_p(a_i c_j - a_j c_i) \\
& + \tau_1(D_j^a c_i - D_i^a c_j) + \tau_2(D_j^c a_i - D_i^c a_j) + \tau_3 a_p D_p^a(a_i c_j - a_j c_i) \\
& + \tau_4 c_p D_p^c(a_i c_j - a_j c_i) + \tau_5(A_j c_i - A_i c_j + C_j a_i - C_i a_j).
\end{aligned} \tag{3.17}$$

For later use it is convenient to express the intrinsic torque due to the latter as

$$e_{ijk}\tilde{t}^{ss}_{kj} = e_{ijk}(a_j\tilde{g}^a_k + c_j\tilde{g}^c_k), \tag{3.18}$$

where

$$\tilde{g}^a_i = -2(\lambda_1 D^a_i + \lambda_3 c_i c_p D^a_p + \lambda_4 A_i + \lambda_6 c_i c_p A_p + \tau_2 D^c_i$$

$$+ \tau_3 c_i a_p D^a_p + \tau_4 c_i c_p D^c_p + \tau_5 C_i), \tag{3.19}$$

$$\tilde{g}^c_i = -2(\lambda_2 D^c_i + \lambda_5 C_i + \tau_1 D^a_i + \tau_5 A_i). \tag{3.20}$$

Amongst other things this allows one to rewrite the viscous dissipation inequality as

$$\tilde{t}^s_{ij} D_{ij} - \tilde{g}^a_i A_i - \tilde{g}^c_i C_i \geq 0, \tag{3.21}$$

which imposes restrictions upon the various viscous coefficients.

Finally, it is useful to note that Equation (3.3) is now equivalent to the equations

$$\left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j} - \frac{\partial W}{\partial a_i} + \tilde{g}^a_i + G^a_i + \lambda a_i + \mu c_i + e_{ijk}\beta_{k,j} = 0,$$

$$\left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j} - \frac{\partial W}{\partial c_i} + \tilde{g}^c_i + G^c_i + \varkappa c_i + \mu a_i = 0, \tag{3.22}$$

and further that Equation (3.2) can be recast as

$$\rho \dot{v}_i = \overline{F}_i - \tilde{p}_{,i} + \tilde{g}^a_k a_{k,i} + \tilde{g}^c_k c_{k,i} + \tilde{t}_{ij,j}, \tag{3.23}$$

where

$$\overline{F}_i = F_i + G^a_k a_{k,i} + G^c_k c_{k,i}, \quad \tilde{p} = p + W, \tag{3.24}$$

this being the dynamic counterpart of the result, Equation (2.16).

Flow Alignment

Here we consider two relatively simple flow configurations, both selected on the grounds that the flow considered is unlikely to disturb the initial planar layer structure. The first envisages a smectic between parallel plates with the layers everywhere parallel to the plates, one of which is set in motion to create simple shear flow. The second has the layers normal to the plates, and again simple shear is generated by moving one plate in a direction parallel to the layers. Both flows produce somewhat unexpected results.

For a preliminary investigation we ignore the influence of the plates upon alignment, and therefore for the first configuration consider a velocity and director fields with Cartesian components

$$\mathbf{v} = (kz, 0, 0), \quad \mathbf{a} = (0, 0, 1), \quad \mathbf{c} = (\cos\varphi, \sin\varphi, 0), \tag{3.25}$$

where k is a positive constant, φ is a function solely of time, and the z-axis is normal to the plates. In this event Equations (3.22) and (3.23) reduce finally to

$$2\lambda_5\dot{\varphi} + k(\tau_5 - \tau_1)\sin\varphi = 0, \tag{3.26}$$

the remaining equations being satisfied by a suitable choice of the unknown multipliers. Since the Inequality (3.21) implies that λ_5 is positive, it follows that

$$\varphi \rightarrow 0 \text{ if } \tau_5 > \tau_1, \quad \text{and} \quad \varphi \rightarrow \pi \text{ if } \tau_1 > \tau_5. \tag{3.27}$$

This result is a little surprising in so far as it is necessary to include so many terms in the viscous stress to produce the result one anticipates. For example, if one replaces the symmetry condition (Equation (2.4)) by one which allows an independent change of sign in either director, this would lead to a loss of flow alignment.

For the second configuration described above, choose

$$\mathbf{v} = (kz, 0, 0), \quad \mathbf{a} = (0, 1, 0), \quad \mathbf{c} = (\cos\varphi, 0, \sin\varphi), \tag{3.28}$$

where again k is a positive constant and φ a function of time. With this choice the equations ultimately yield

$$2\lambda_5\dot{\varphi} + k(\lambda_5 + \lambda_2\cos 2\varphi) = 0, \tag{3.29}$$

which may or may not produce flow alignment depending upon the relative magnitudes of λ_2 and λ_5. However, in either event the shear stress producing the flow is not parallel to the flow direction, there being a non-zero transverse component. When surface effects are included and the alignment has a spatial dependence, this must lead to transverse flow which is unexpected.

4. CONCLUDING REMARKS

While the equations in the above theory are perhaps rather more complex than one might at first wish, they do appear to be the minimum necessary to provide a credible model. Nonetheless a surprising amount of progress has been possible, particularly in terms of solutions for static configurations, this creating confidence as to the relevance of the theory. Moreover, there are already first indications of the value of such a continuum theory in that its preliminary predictions point to novel effects encouraging related experiments. In this latter respect, it would certainly be useful to have some evidence as to how flow and external fields affect the layer structure in smectics through careful experiments with optical monitoring. Perhaps the existence of a plausible theory will stimulate new observations, which in turn will encourage relevant calculations.

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